M.Sc (Sem 03) Unit-03
paper 11

Complex linear spaces and normed linear spaces

Theorem:- Let N be a normed linear space and x_0 a ~~non-empty~~ zero vector in N. There exists a functional F in N^* such that

$$F(x_0) = \|x_0\| \text{ and } \|F\| = 1.$$

In particular, if $x \neq y$ $(x, y \in N)$, then there exists an $f \in N^*$ such that $f(x) \neq f(y)$

Proof:— Let $M = \{\alpha x_0\}$ be the linear subspaces of N spanned by x_0. Define f_0 on M by $f_0(\alpha x_0) = \alpha \|x_0\|$. We show that f_0 is a functional on M such that $\|f_0\| = 1$.

f_0 is linear.

Let $y_1, y_2 \in M$ so that $y_1 = \alpha x_0$, $y_2 = \beta x_0$ for some scalars α and β. If γ, δ are any scalars, then

$$f_0(\gamma y_1 + \delta y_2) = f_0(\gamma \alpha x_1 + \delta \beta x_0) = f_0((\gamma\alpha + \beta\delta)x_0)$$
$$= (\gamma\alpha + \delta\beta)\|x_0\| \text{ by def. of } f_0 = \gamma\alpha\|x_0\| + \delta\beta\|x_0\|$$
$$= \gamma f_0(\alpha x_0) + \delta f_1(\beta x_0) = \gamma f_0(y_1) + \delta f_0(y_2)$$

f_0 is bounded.

Let $y = \alpha x_0 \in M$ so that $\|y\| = \|\alpha x_0\| = |\alpha|\,\|x_0\|$. Now $|f_n(y)| = |f_0(\alpha x_0)| = |\alpha\|x_0\||$
$$= |\alpha|\,\|x_0\| = \|y\| < |y|$$

Hence f_0 is bounded. It follows that f_0 is a functional on M.

Further, $\|f_0\| = \sup\{|f_0(y)| : y \in M, \|y\| = 1\}$.
$$= \sup\{\|y\| : y \in M, \|y\| \leq 1\} = 1.$$

Also $f_0(x_0) = \|x_0\|$ be definition of f_0 (choose $\alpha = 1$). Hence by Hahn-Banach theorem f_0 can be extended to a norm preserving functional $F \in N^*$ so that

$$F(x_0) = f_0(x_0) = \|x_0\| \text{ and } \|F\| = \|f_0\| = 1.$$

In the particular case, since $x \neq y$, $x - y \neq 0$ and so we know that there exists an $f \in N^*$ such that

$$f(x-y) = \|x-y\| \neq 0 \Rightarrow f(x) - f(y) \neq 0 \Rightarrow f(x) \neq f(y)$$

This shows that N^* separates vectors in N.

Theorem: Let M be a closed linear subspace of a normed linear space N and x_0 a vector not in M. Then there exists a functional F in N^* such that

$$F(M) = \{0\} \text{ and } F(x_0) \neq 0$$

Proof — Consider the natural map

$$\phi : N \to N/M : \phi(x) = x+M$$

~~As shown~~ ϕ is a continuous linear transformation and if $m \in M$, then $\phi(m) = m+M = 0$

$$\phi(M) = \{0\} \quad \text{——— (1)}$$

Also since $x_0 \notin M$, we have

$$\phi(x_0) = x_0 + M \neq 0 \quad (\text{i.e. } \neq \text{ zero vector } M \text{ of } N/M)$$

Hence we ~~know that~~ have there exists a functional $f \in (N/M)^*$

such that $f(x_0+M) = \|x_0+M\| \neq 0$ ——— (2)

$[\because x_0 + M \neq \text{Zero Vector}]$

We now define F by $F(x) = f(\phi(x))$. Then F is a linear functional on N with the desired properties as shown below.

F is linear.

$$F(\alpha x+\beta y) + f(\phi(\alpha x+\beta y)) + f((\alpha x+\beta y)+M)$$

$$= f(\alpha(x+M) + \beta(y+M))$$

$$= \alpha f(x+M) + \beta f(y+M) \quad [\because f \text{ is linear on } N/M]$$

$$= \alpha f(\phi(x)) + \beta f(\phi(x)) = \alpha F(x) + \beta F(y)$$

F is bounded.

$$|F(x)| = |f(\phi(x))| \leq f\,\|f\|\,\|\phi(x)\|$$

$$\leq \|f\|\,\|\phi\|\,\|x\| \leq \|f\|\,\|x\|$$

$$[\because \|\phi\| \leq 1]$$

Since f is bounded (being a functional), it follows from the above inequality that F is bounded. Thus F is a functional on N i.e. $F \in N^*$. Further if $m \in M$, then $F(m) = f(\phi(m)) = f(0) = 0$ $[\because \phi(m) = 0 \text{ by (1)}]$

And $F(x_0) = f(\phi(x_0)) = f(x_0+M) \neq 0$ by (2).

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